Comparing numerical simulations of free surface flows using non-hydrostatic Navier-Stokes and Boussinesq equations.

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ABSTRACT: The numerical solutions of three-dimensional, non-hydrostatic Navier-Stokes equations are compared to the ones obtained with Boussinesq equations in the case of a solitary wave in a rectangular channel. The solution procedures used in Telemac-2D for Boussinesq and in Telemac-3D for Navier-Stokes are presented and the numerical results are compared, focusing on the computational time. Depending on the number of mesh levels used in 3D, the solution of full Navier-Stokes equations can be computationally more advantageous than applying the 2D Boussinesq equations.

1 INTRODUCTION

The software system TELEMAC, based on Element By Element techniques, addresses free surface and underground flows. In its present development stage, the system includes the Saint-Venant or Shallow Water Equations in 2 dimensions solved using a computer program called Telemac-2D, Navier-Stokes equations in 3 dimensions with a free surface (Telemac-3D, until recently only with hydrostatic assumption), and also Mild Slope equations, Wave Action equations, water quality models, sediment transport equations in 2 and 3 dimensions, Richard's equations in 2 and 3 dimensions.

Recently, improvements of the system have led, on one hand, to a new option in Telemac-2D allowing solution of Boussinesq equations, and on the other hand to a non-hydrostatic option in Telemac-3D. First, we briefly describe Telemac-2D and give an insight into the solution procedure for Boussinesq equations. Then the new algorithm for the non-hydrostatic option in Telemac-3D will be presented in detail and possible variants discussed. In the fourth paragraph both programs are compared using the example of a solitary wave.

2 SHORT DESCRIPTION OF TELEMAC-2D

TELEMAC-2D solves the primitive shallow water equations written in the non-conservative depth-velocity form. Mass-conservative schemes such as Streamline Upwind Petrov Galerkin (S.U.P.G., see Brooks et al. 1982) may be used to solve the continuity equation. The main features of TELEMAC 2D are the use of Cartesian or spherical coordinates, the possibility of dealing with subcritical or supercritical regimes, a number of source terms such as wind stress, atmospheric pressure and Coriolis force, an equation of a tracer concentration, tidal flats treatment and various types of boundary conditions including free slip and incident waves.

Applications at LNH concern the computation of floods in rivers, thermal plumes, impact of intakes and outfalls, dam-break floodwave simulations. The results, velocity field and depth, are used by other modules of the system in order to deal with water quality, sediment transport, and also for coupling with wave models.

The following equations are simultaneously solved:

Continuity: $$\frac{\partial h}{\partial t} + \vec{u}\cdot\overrightarrow{grad}(h) + h\,div(\vec{u}) = 0 \qquad (1)$$

Momentum:

$$\frac{\partial u}{\partial t} + \vec{u}.\overrightarrow{grad}(u) = -g\frac{\partial Z}{\partial x} + F_x + \frac{1}{h}div\left(h\nu_t\,\overrightarrow{grad}\,u\right) \qquad (2)$$

$$\frac{\partial v}{\partial t} + \vec{u}.\overrightarrow{grad}(v) = -g\frac{\partial Z}{\partial y} + F_y + \frac{1}{h}div\left(h\nu_t\,\overrightarrow{grad}\,v\right) \qquad (3)$$

where h is the depth, u and v the components of horizontal velocity, Z is the free surface, F_X and F_Y are source terms (friction, etc.), and ν_t is the turbulent viscosity. Detailed explanations of the numerical algorithm are given in Hervouet et al. (1996). The Boussinesq equations are basically similar to Shallow Water Equations and are also derived by depth-averaging the Navier-Stokes equations, with the difference that the pressure is not assumed to be hydrostatic. The vertical velocity is considered to be first order with respect to ε and second order with respect to μ, where ε is $\frac{a}{h}$, the wave amplitude divided by the depth, and μ is $\frac{h}{l}$, the water depth divided by the wavelength. Compared to the pure Navier-Stokes equations, this condition will strongly limit the application scope.

The continuity equation of Shallow Water Equations is left unchanged by the Boussinesq assumptions, but the theory yields new terms in the right hand side of the momentum equation, in the form:

$$-\frac{H_0^2}{6}\vec{\nabla}\left(\nabla\left(\frac{\partial\vec{u}}{\partial t}\right)\right) + \frac{H_0}{2}\vec{\nabla}\left(\nabla\left(H_0\frac{\partial\vec{u}}{\partial t}\right)\right) \qquad (4)$$

where H_0 is a reference depth. After testing several models and options (see Hervouet et al., 1999) the Boussinesq term has been discretised and added in the Telemac-2D algorithm in a straightforward way. Due to the fact that this term contains the coupled unknown horizontal velocity components, it is implicit. Without Boussinesq terms, one eventually solves a linear system coupling the components of velocity u and v and the depth h, of the form A X = B, with:

$$A = \begin{pmatrix} AM1 & BM1 & BM2 \\ CM1 & AM2 & 0 \\ CM2 & 0 & AM3 \end{pmatrix} \quad (5)$$

$$X = \begin{pmatrix} \delta H \\ U \\ V \end{pmatrix} \quad (6) \qquad \text{and} \qquad B = \begin{pmatrix} CV1 \\ CV2 \\ CV3 \end{pmatrix} \quad (7)$$

where AM1, AM2, AM3 are square matrices, BM1, BM2, CM1, CM2 may be rectangular matrices if velocity and depth are discretised in a different way, and CV1, CV2 and CV3 are right hand side terms. δH is the vector made of the increment of depth in one time-step, i.e. $h^{n+1} - h^n$, for every node of the mesh. Although this approach yields a large system of equations to be solved, it avoids problems due to fractional steps or modified equations such as the wave equation. When Shallow Water Equations are used, u and v are not coupled (this is due to the use of weak boundary conditions for ensuring $\vec{u}.\vec{n} = 0$ on the solid boundaries). With the extra Boussinesq terms, the coupling of u and v yields two additional matrices in the block A, but the principle of the solution procedure is not changed. However, the conditioning of the linear system may be downgraded by the extra Boussinesq terms. For small values of $\frac{a}{h}$, the extra computational effort is only a few percents larger than pure Shallow Water Equations. In difficult cases the price may be ten times higher. As a matter of fact, with second order space derivative of a derivative in time, Boussinesq equations are comparable to Helmholtz equations, and the linear systems they lead to are equally difficult to solve with iterative methods. Generalised Minimum RESidual with diagonal preconditioning (Hughes et al. 1987, Saad et al. 1983) seems to be the most efficient technique for this option. The algorithm appeared to be robust and could even cope with tidal flats or dry zones, which is surprising due to the fact that Boussinesq equations cannot reproduce properly drying zones (the reference depth looses its meaning).

Other numerical algorithms have been tested in the hope of decreasing the computational cost of the Boussinesq option, but all other techniques make use of approximations or assumptions which may be invalid. For example, they sometimes assume a rotational-free 2-dimensional velocity field. This assumption leads to instabilities in applications at LNH, in a study of waves induced by landslides in reservoirs, due to high velocities on complex solid boundaries, where probably the rotational is not nil. Such drawbacks do not appear in most academic test cases in simple geometries.

Decoupling u and v has been tried as well, and also led to instabilities. Our conclusion is that Boussinesq equations are intrinsically difficult due to their Helmholtz character, that robustness has a high price, and that most well known simplifications cannot be used in real-life applications.

3 TELEMAC-3D AND THE NON-HYDROSTATIC ALGORITHM.

3.1 Hydrostatic option

TELEMAC-3D solves the Navier-Stokes equations with a free surface, together with the advection-diffusion equations of temperature, salinity and any other variables. Density effects, wind stress on the free surface, heat exchange with the atmosphere and the Coriolis force are included.

The space discretization is realised using prismatic elements. The quadrangular sides of the prisms are vertical. The vertical 2D projection of the mesh is made of triangles, and we only need to mesh the 2D horizontal domain and then duplicate it along the vertical to cover the 3D domain.

In the basic version, the pressure is assumed hydrostatic and the free surface is computed by solving equations similar to the shallow-water equations. The equations read:

Continuity:

$$div(\vec{U}) = 0 \quad (8)$$

Momentum:

$$\frac{\partial u}{\partial t} + \vec{U}.\overrightarrow{grad}(u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + div(\nu_t \overrightarrow{grad}(u)) + f_x \quad (9)$$

$$\frac{\partial v}{\partial t} + \vec{U}.\overrightarrow{grad}(v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + div(\nu_t \overrightarrow{grad}(v)) + f_y \quad (10)$$

Where ρ_0 is the reference density, ν_t the turbulent viscosity, and f_x and f_y are source terms such as the Coriolis force.

The vertical velocity does not appear in the momentum equations and will be deduced from the continuity equation. The hydrostatic pressure is written:

$$p = \rho\, g\, (S-z) + \rho\, g \int_z^S \frac{\Delta\rho}{\rho}\, dz \quad (11)$$

where the second term takes into account the buoyancy effects due to temperature or salinity.

According to an operator splitting approach, the solution procedure contains three steps: advection step, diffusion step and free surface-continuity-pressure step, summarised by the following equation (where f stands for u or v):

$$\frac{\partial f}{\partial t} = \frac{f^{n+1} - f^d}{\Delta t} + \frac{f^d - f^a}{\Delta t} + \frac{f^a - f^n}{\Delta t} \quad (12)$$

The advection step is $\frac{f^a - f^n}{\Delta t} + \vec{U}.\overrightarrow{grad}(f) = 0$. It can be solved, depending on the chosen option and due to its hyperbolic character, by the method of characteristics, by S.U.P.G., or by distributive schemes such as the PSI scheme (Paillère 1995).

The diffusion step is $\frac{f^d - f^a}{\Delta t} = div(\nu_t \overrightarrow{grad}(f))$. In the pressure and free surface step the depth-averaged values of the horizontal velocity field are applied using parts of the Telemac-2D algorithm. Finally, the vertical velocity w is obtained from the 3-dimensional continuity equation. This method is valid only when the vertical velocity remains small, i.e. within the assumptions of hydrostatic pressure distribution.

To overcome the problems of free surface movements, the equations are actually solved in a σ-transformed mesh, where the vertical coordinate varies in the range [0,1].

3.2 non hydrostatic option

After Jacek Jankowski's PhD at the University of Hannover (ref. [2]), a non-hydrostatic option for Navier-Stokes equations has been added to Telemac-3D. Following the already existing hydrostatic algorithm in the hydrostatic option described above, the operator splitting approach has been used. As previously, the advection (a hyperbolic step) and diffusion (parabolic step) in the momentum equations are solved first.

Then a Poisson Pressure Equation (PPE) (an elliptic step), followed by a velocity projection, is solved to ensure a divergence-free velocity field.

The basic idea of the method is a decomposition of the pressure into the hydrostatic pressure p_h and the hydrodynamic pressure denoted π: $p = p_h + \pi$. In the first phase (which can be itself decomposed into several steps), only the hydrostatic pressure is taken into account and we solve:

$$\frac{\tilde{u} - u^n}{\Delta t} + \vec{U}.\overrightarrow{grad}(u) = -\frac{1}{\rho_0}\frac{\partial P_h}{\partial x} + div(\nu_t \overrightarrow{grad}(u)) + f_x \quad (13)$$

$$\frac{\tilde{v} - v^n}{\Delta t} + \vec{U}.\overrightarrow{grad}(v) = -\frac{1}{\rho_0}\frac{\partial P_h}{\partial y} + div(\nu_t \overrightarrow{grad}(v)) + f_y \quad (14)$$

$$\frac{\tilde{w} - w^n}{\Delta t} + \vec{U}.\overrightarrow{grad}(w) = -\frac{1}{\rho_0}\frac{\partial P_h}{\partial y} - \frac{\rho}{\rho_0} + div(\nu_t \overrightarrow{grad}(w)) \quad (15)$$

In the next step the hydrodynamic pressure is taken into account to ensure the continuity equation and yield a divergence free velocity field: $\nabla^2(\pi) = \frac{\rho_0}{\Delta t} div(\vec{\tilde{U}})$, where $\vec{\tilde{U}}$ is the vector of components $(\tilde{u},\tilde{v},\tilde{w})$. For solving this step the iterative technique of Stabilised Biconjugate Gradient with Crout preconditioning is applied (ref. [4]). Surprisingly, this method for solving linear equations appears to be more efficient than GMRES. It should be noted that Crout preconditioning is very appropriate for linear systems stemming from Laplacians.

The final velocity is obtained by projecting the $(\tilde{u},\tilde{v},\tilde{w})$ field using the hydrodynamic pressure gradient:

$$\vec{U} = \vec{\tilde{U}} - \nabla(\pi) \quad (16)$$

Finally, the free surface is updated. This step is one of the main difficulties. Two options have been tried:

- Using a kinematic boundary condition for the free surface, which consists of moving the boundary according to the velocity at the free surface. This classical solution was found to be less robust than the following one.

- Using the depth-averaged continuity equation. As a matter of fact, this equation generally used for Shallow Water Equations is also perfectly valid here, but a depth-averaged velocity must be computed. Assuming a constant bottom, we solve:

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}\left(\int_{bottom}^{S} u\, dz\right) + \frac{\partial}{\partial y}\left(\int_{bottom}^{S} v\, dz\right) = 0 \quad (17)$$

which is written in the form:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) + \frac{\partial}{\partial y}(h\bar{v}) = 0 \quad (18)$$

where $\bar{u}$ and $\bar{v}$ are the depth-averaged components of the horizontal velocity. In this equation the final velocity is applied in order to obtain the final depth. This equation was already thoroughly studied for implementing Telemac-2D and also during our research work on Boussinesq equations. Material from Telemac-2D could then be re-used and an implicit scheme with the already cited Streamline Upwind Petrov Galerkin technique proved to be efficient. Because the velocities are known, the problem is linear, the divergence term in the equation can be treated properly to ensure a perfect mass conservation.

Knowing the new free surface elevation, the mesh is updated and a new time-step can be performed. The advection and diffusion steps of the first phase can also be done with a σ-mesh.

To sum up, compared to classical Navier-Stokes solvers for internal flows, there are now two key-points in the algorithm:

1) The pressure is split into a hydrostatic pressure and a dynamic pressure. The gradient of the hydrostatic pressure is added as a source term in the advection-diffusion step, and only the dynamic pressure is thus used in the PPE step. As a consequence, this step tends to become trivial when a flow is nearly hydrostatic.

2) The free surface is obtained using a depth-averaged continuity equation. Already existing material for solving Shallow Water Equations can be conveniently re-used.

4 APPLICATIONS

4.1 Solitary wave

A series of tests has been performed with a solitary wave in a 600 m long rectangular channel, with flat bottom. Depth h is 10 m and the wave height H is 2 m. It is well known that the hydrostatic assumption is not applicable for describing such a wave. Either Shallow Water equations or hydrostatic 3D Navier-Stokes equations lead to an artificial breaking. The solitary wave must propagate without changing its shape and celerity, yielding an ideal case for a model verification with a clear criterion. However, only approximate mathematic descriptions are available for solitary waves, making comparisons of the numeric to analytic solutions problematic.

This probably explains the wavelets in the wake, which are observed applying both models. As a matter of fact, we have chosen for the initial condition the second approximation to a solitary wave given by Laitone in 1960 (Laitone, 1960):

$$u = \sqrt{gh}\,\frac{H}{h}\,sech^2\left[\sqrt{\frac{3H}{4h^3}}(x-ct)\right] \quad (19)$$

$$w = \sqrt{3gh}\left(\frac{H}{h}\right)^{3/2}\frac{z}{h}\,sech^2\left[\sqrt{\frac{3H}{4h^3}}(x-ct)\right]\tanh\left[\sqrt{\frac{3H}{4h^3}}(x-ct)\right] \quad (20)$$

$$\eta = h + H\,sech^2\left[\sqrt{\frac{3H}{4h^3}}(x-ct)\right] \quad (21)$$

u and w are respectively the horizontal (x axis) and vertical components of velocity. c is the celerity of the wave, equal to $\sqrt{g(H+h)}$, and η is the free surface elevation. sech is the hyperbolic secant 1/ch(x). In this solution the pressure can be considered hydrostatic at the second order with respect to H/h. As the initial wave extends from $-\infty$ to $+\infty$, the position of the maximum was chosen sufficiently far (80 m) from the boundary behind the wave.

Despite the fact that this wave is not exactly described with our two sets of equations, it behaves reasonably well with both of them. The non-hydrostatic option yields a behaviour of the wave comparable with what is obtained with a Boussinesq solver. For both models the time step is 0.1 s and the duration is 40 s. Figure 1 displays the vertical cross-section of the rectangular channel, with the solitary wave at different time steps. This result has been obtained using the 3D model with only 3 planes on the vertical, i.e. two layers of prisms.

Computer time is only 60% of what is required with our Boussinesq solver. Table 1 summarises the computational times obtained with different equations and different number of layers in the case of 3D. Times are given for a HP 712/80 workstation (133 MHz). The number of triangles in the mesh is 7206, the number of prisms in the 3D case with 11 levels is thus for example 72060.

Equations	Computer time
Shallow Water Equations (not relevant to the case)	752 s
Boussinesq	6114 s
3D hydrostatic 11 levels (not relevant to the case)	3969 s
3D non hydrostatic 11 levels	10176 s
3D non hydrostatic 06 levels	5008 s
3D non hydrostatic 03 levels	3712 s

Table 1: computational time of different equations

Shallow Water Equations and 3D hydrostatic are given in the table for comparison but are of course not relevant to the case of a solitary wave. They both give the same free surface elevation, which tends to a breaking of the solitary wave. The effect of adding the Boussinesq terms in the Shallow Water Equations is tremendous and the computer time rises from 752 s to 6114 s. Compared to this result, the 3D computation with 3 levels appears to be much more attractive. The main time-consuming difficulties are, on Navier-Stokes side, the Poisson Pressure Equation, on Boussinesq side, the Helmholtz-like equations stemming from the extra terms added by Boussinesq in the Saint-Venant equations. In both cases matrices of the linear systems are mostly Laplacian matrices, which explains the same order of difficulties.

4.2 Lock exchange flow

In order to illustrate the feasability of the non-hydrostatic model for other flow types as well, the intrusion of a salt water wedge into a fresh water channel, after opening a lock gate, is additionally presented. In this well known test-case, applying the hydrostatic assumption for a gravitational current yields a wrong solution. The case is schematised as follows: two fluids with different densities are confined in a rectangular basin with impermeable walls and, at the initial time, an horizontal free surface. The left part of the basin is filled by the fluid of higher density. An interfacial wave is created and the denser fluid propagates on the right, along the bottom. The hydrostatic solution shows a rectangular pattern which is not observed in experiments, be it in nature or in laboratories. The non-hydrostatic solution is validated by numerous measurements (see Barr et al. 1967, Yih 1980). Figure 2 compares the hydrostatic (above) and non-hydrostatic (below) results, with salinity iso-lines (every 3.8 g/l) and velocity field. The density difference is obtained with a salinity: 38 g/l for the denser fluid and 0 g/l for the other. The much better results of the non-hydrostatic version open the possibility of refined flow modelling in the field of 3-dimensional thermal plumes, or in salt-wedge studies in estuaries.

5 CONCLUSION

In this paper we have presented and compared a 2-dimensional and a 3-dimensional approach for the numerical simulation of waves. A robust algorithm can be built for solving Boussinesq equations in finite elements but tends to be computationally expensive and requires more expertise compared to Shallow Water Equations. Boussinesq equations have severe theoretical assumptions limiting their application to a narrow range of wave amplitude and length. Full 3D non-hydrostatic Navier-Stokes solvers are now available for the simulation of waves and the present comparison shows they could replace Boussinesq solvers in the near future. Our Boussinesq solver has been designed for difficult industrial cases and, due to implicitness and coupling of depth and velocity, is perhaps not optimised for simpler cases. However, even if well known technical problems and considerable computer times may be expected for the entirely 3D Navier-Stokes solvers, it seems reasonable to concentrate the efforts on this promising approach, rather than undertaking an improvement of approximated equations which the general trend of progress will eventually outdate.

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